# Some Notes on Low Emittance Flat Beam Generation

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February 22, 2006

**Abstract**

The recent successful generation of flat electron beams by Edwards, *et al.*,[1], spurred by the proposition set forth by Derbenev, Brinkmann, and Flottmann,[2] leads to the anticipation of very bright electron sources for use in a future linear collider. Horizontal-vertical emittance ratios of 100:1 or better have been generated at Fermilab. The emittances achieved are within a factor of two of the linear collider emittance specifications, though at a lower beam intensity. To make further improvements in overall beam brightness, a reduction in the final horizontal emittance clearly would be beneficial. The present proposition is to generate a spiral pattern onto a photo-cathode and pass the resulting beam through a flat-beam transformation. While the vertical phase space area becomes very small, the horizontal phase space will result in a corresponding spiral pattern. Using a magnetic focusing channel with suitable non-linear fields, this spiral pattern can be "unwound" in order to group the particles into a smaller region of horizontal phase space. While space charge effects at the cathode still need to be addressed, the possibility of small *and* flat beams exists. Increases in overall beam brightness by factors of 5 or so may be possible.

## 1 Review of Flat Beam Generation

The principle behind the "flat beam transformation"[2] is to generate a beam of particles submerged in a solenoidal magnetic field, initially all traveling parallel to the field direction which. Upon exit of the solenoid, the beam acquires angular momentum about the solenoid axis. After a suitable transformation using skew quadrupoles, the vertical trajectories of all the particles in the "ideal" beam become identical (zero vertical phase space emittance) and the horizontal phase space will be defined by the original physical spot size within the solenoid. Below we illustrate the process using a simple beam transformation as an example. In the original Fermilab demonstration experiment the optical system used three quadrupoles to generate appropriate phase advance in each degree of freedom and eventually acquired emittance ratios of 100:1.[3]

### 1.1 The Transformation

To see how the flat beam transformation works, imagine an electron being generated on a cathode surface and emerging with trajectory

$$\vec{X} = \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix} = \vec{X}_0 = \begin{pmatrix} x_0 \\ 0 \\ y_0 \\ 0 \end{pmatrix} = \begin{pmatrix} r_0 \cos\theta_0 \\ 0 \\ r_0 \sin\theta_0 \\ 0 \end{pmatrix} \tag{1}$$

where the transverse dimensions are relative to an ideal trajectory along the axis of symmetry of a solenoid field. Upon exit of the solenoid, the trajectory will be transformed according to

$$\vec{X}_s = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -k & 0 \\ 0 & 0 & 1 & 0 \\ k & 0 & 0 & 1 \end{pmatrix} \vec{X}_0 = \begin{pmatrix} x_0 \\ -k\, y_0 \\ y_0 \\ k\, x_0 \end{pmatrix}. \tag{2}$$

where $k \equiv B_s/2(B\rho)$, $B_s$ is the central field value of the solenoid, and $B\rho = p/e$ is the particle's magnetic rigidity. The particle next enters a focusing section, which has been rotated by 45°, *i.e.*, made of skew quadrupoles. The section has a "vertical" transformation which is 90° out of phase with the "horziontal" transformation. For simplicity, assume the "horizontal" transformation is unity, while the "vertical" transformation has a phase advance of 90°, and a periodic amplitude function $\beta_0 \equiv 1/k$. In matrix form, the total transformation is

$$\vec{X}_f = R^{-1} M_c R \vec{X}_s,$$

where

$$R = \frac{\sqrt{2}}{2} \cdot \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix},$$

which is a rotation of 45° of the coordinates, and

$$M_c = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \beta_0 \\ 0 & 0 & -1/\beta_0 & 0 \end{pmatrix}$$

corresponding to the unrotated FODO channel described above. Carrying out the matrix multiplications, we arrive at the trajectory leaving the rotated FODO channel:

$$\vec{X}_f = \begin{pmatrix} x_0 - y_0 \\ k(x_0 + y_0) \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} r_0(\cos\theta_0 - \sin\theta_0) \\ kr_0(\cos\theta_0 + \sin\theta_0) \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} r_c \cos\theta_c \\ kr_c \sin\theta_c \\ 0 \\ 0 \end{pmatrix}. \tag{3}$$

From the last relationship we see that the particle will lie on a circle of radius $r_c = \sqrt{2}\, r_0$ in $x, x'/k$ phase space, and will lie at an angle $\theta_c$ in this phase space, where $\theta_c = \theta_0 + \pi/4$. The result is illustrated in Figure 1. Naturally, a realistic beam will have a thermal emittance as it emerges from the cathode and so the vertical emittance will not be zero and there will be a thermal "noise" term to include in the horizontal phase space coordinates. For this discussion the thermal emittance will be ignored.

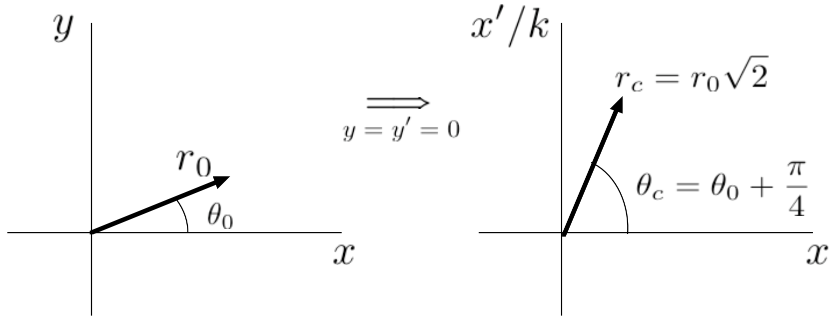


Figure 1: A particle emerging from the solenoid with real-space coordinates $r_0, \theta_0$ will end up with zero vertical emittance and a horizontal phase space coordinate of $\sqrt{2}\, r_0, \theta_0 + \pi/4$.

### 1.2 Resulting rms Emittance from Uniform Cathode Illumination

Consider an ideal initial distribution of particles traveling parallel to the central solenoid field and uniformly distributed transversely within a radius $a/\sqrt{2}$ of the central axis. Following the "flat beam transformer" the resulting horizontal phase space distribution will be uniform within a circle of radius $a$ and will have variance

$$\sigma_x^2 \equiv \langle x_c^2 \rangle = \frac{1}{2}\langle r_c^2 \rangle = \frac{1}{2}\int_0^a r^2 r dr \;/\; \int_0^a r dr = (1/2)(a^4/4)/(a^2/2) = a^2/4.$$

The rms emittance would thus be $\epsilon = \pi a^2/4\beta_0 = \pi k a^2/4$. This result may be of use later in this document for comparisons.

## 2 Reducing the Horizontal Phase Space

In our above model of a perfect flat beam transformer the resulting vertical and horizontal emittances are $\epsilon_y = 0$ and $\epsilon_x = \pi k a^2/4$. We would like to develop a smaller horizontal emittance. One possible method is to form a spiral pattern in the transverse dimensions at the cathode, within the solenoid field, which results in a spiral pattern in the horizontal phase space after the flat beam transformer. Then, by a suitable beam transport channel with nonlinear fields, the spiral pattern can be "unwound" to form a final distribution of lesser overall extent in phase space due to the amplitude dependence of phase advance through the nonlinear system. This is, in a sense, the time reversal of the emittance dilution generated by a steering error upon injection into a nonlinear periodic lattice.

## 2.1 Transforming a Spiral Pattern into Horizontal Phase Space

From the result of Subsection 1.1, a spiral pattern formed at the cathode, say, with particles traveling parallel to the solenoid field, will upon passing through the flat beam transformer result in a similar spiral pattern in the horizontal phase space, scaled by a factor of $\sqrt{2}$ and rotated by an angle of 45°. This is illustrated in Figure 2. By a suitable choice of spiral pattern, and a focusing

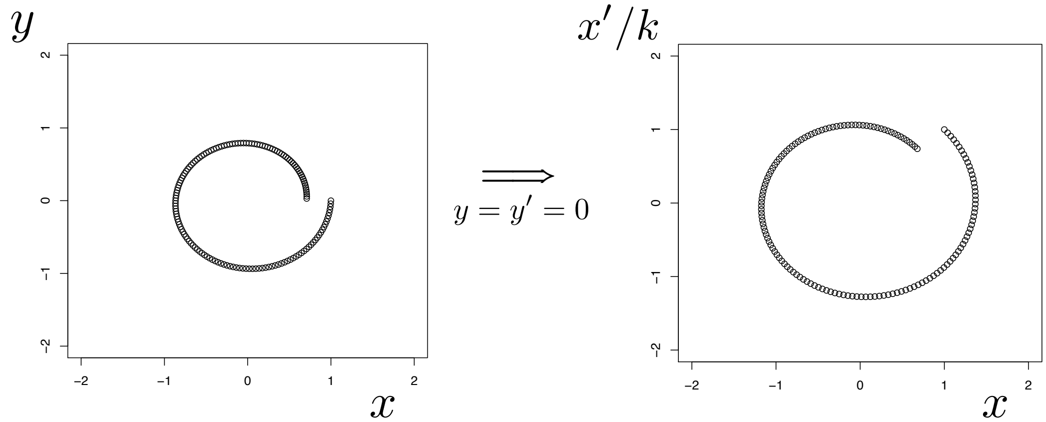


Figure 2: A spiral pattern emerging from the solenoid will obtain zero vertical emittance and a spiral horizontal phase space pattern of rotated by an angle of $\pi/4$ and scaled by $\sqrt{2}$.

channel with appropriate nonlinear fields, the spiral pattern can be transformed into a straight line in phase space.

## 2.2 Unrolling the Spiral Phase Space

We wish to have our spiral phase space pattern be the initial condition at the entrance to a nonlinear focusing channel. Imagine a FODO lattice which contains along its length a systematic nonlinear field which generates an amplitude dependent transverse phase advance, such as that produced by a series of octupoles. The phase advance $\psi$, or more specifically, the tune $\nu = \psi/2\pi$ would vary like

$$\nu = \nu_0 + \nu'' r^2$$

where $\nu_0$ is the zero amplitude tune of a single FODO cell. The non-linear de-tuning coefficient is given by[5]

$$\nu'' = \frac{3}{16\pi} \frac{\hat{\beta}}{B\rho} \int_0^\ell \left( \frac{\beta(s)}{\hat{\beta}} \right)^2 \frac{B'''(s)}{6} ds \tag{4}$$

where $\hat{\beta}$ is the amplitude function at the point corresponding to amplitude $r$ above, and $\beta(s)$ is the amplitude function which varies over the octupole field found within each FODO cell of length $\ell$.

If we choose an initial phase space pattern beginning at radius $r_1$ and spiraling through an angle of $2\pi$ radians into radius $r_2$, then the tune difference between the points at each end of the pattern would be

$$\Delta\nu = \nu''(r_2^2 - r_1^2).$$

After $N \approx 1/\Delta\nu$ FODO cells, the two end points would lie along a single radius in phase space. To ensure that all points along the spiral pattern will lie along the same radius after $N$ cells, we may choose

$$r_0 \sim \sqrt{\theta_0}, \quad \text{or}, \quad \nu \sim \theta_0$$

for our initial phase space distribution.

### 2.3 A Numerical Example

As a simple example, consider an initial phase space distribution which is an infinitesimally thin spiral of functional form

$$r(\theta) = \sqrt{\frac{1}{2}\left[1 + (\theta/2\pi)\right]} \tag{5}$$

in arbitrary units of length. The "tune spread" of the distribution is $\Delta\nu = \nu''(1 - 1/2) = \nu''/2$. If we choose $\nu'' = 0.1$ as an example, then it would take 20 FODO cells for the spiral to turn into a straight line of length $\Delta r = 1 - 1/\sqrt{2} = 0.29$ in our phase space.

Figure 3 shows the result of a simple simulation of the process. In (a) the phase space pattern is given by Eq. 5. The distribution evolves through the FODO channel until, 20 cells later, we arrive at (d), where the phase space is a straight line of length 0.29.

### 2.4 Estimate of Resulting Emittance

The next step in the process would be to use steering elements to center the resulting beam onto the design trajectory of the downstream beam line. If we imagine further nonlinearities, which must be present in any real system, then we would expect our "line" distribution to begin to filament into a final distribution which is cylindrically symmetric in phase space, as depicted in Figure 4. Thus, the final distribution would have a variance given by

$$\sigma_x^2 = \langle x^2 \rangle^{final} = \frac{1}{2}\langle r^2 \rangle^{final} = \frac{1}{2}\,\frac{\int_0^{\Delta r/2} r^2 dr}{\int_0^{\Delta r/2} dr} = \frac{1}{2} \cdot \frac{(\Delta r/2)^3/3}{(\Delta r/2)} = \frac{1}{24}\Delta r^2$$

or, $\sigma_x = 0.204\Delta r$. For $\Delta r = 1 - 1/\sqrt{2}$, $\sigma_x = 0.06$, which is borne out by the simulation. One can imagine giving the initial curve a "thickness" by adding random displacements with rms $\delta$ in $x$ and $x'/k$ to points along the spiral. The final rms size in our example would then be

$$\sigma_x = \sqrt{(0.06)^2 + \delta^2}.$$

If for example the curve had "thickness" $2\delta \approx 0.2$ (20% of the final amplitude of the spiral), then the resulting rms would be $\sigma_x = 0.12$, or a final variance of 0.014. This is depicted in Figure 5.

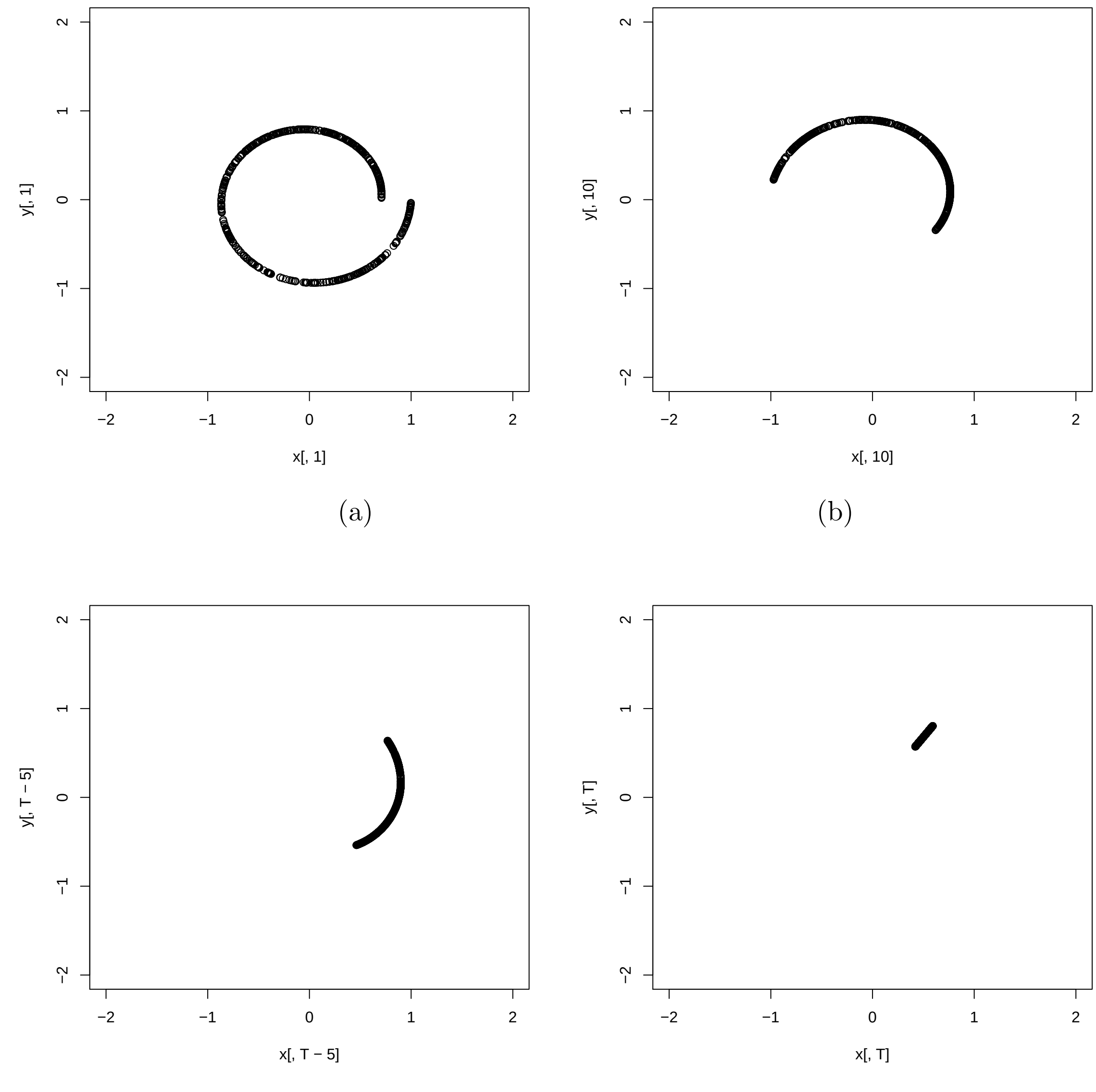


Figure 3: Development of phase space distribution through a nonlinear FODO channel. (a) Initial spiral distribution. (b) After 10 FODO cells with $\nu'' = 0.1$. (c) After 15 cells. (d) After 20 cells.

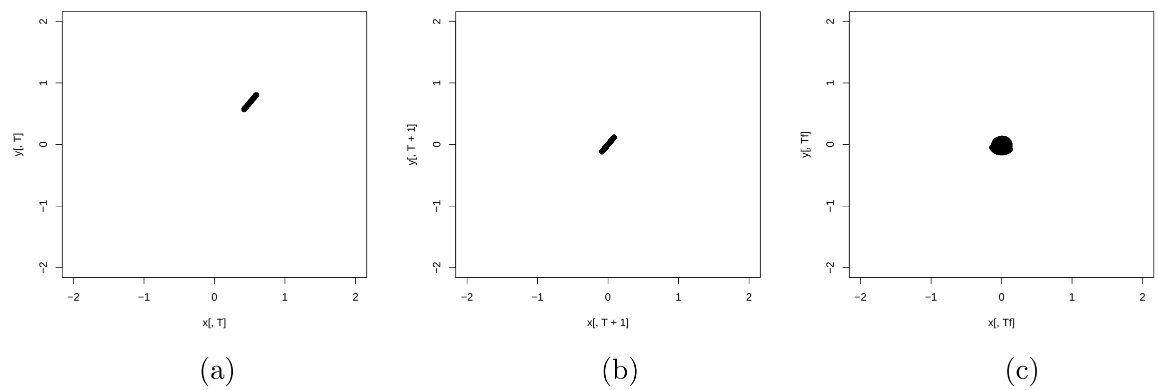


Figure 4: Continuation of phase space evolution. (a) After the 20-cell channel. (b) Recentering using steering elements. (c) After final filamentation.

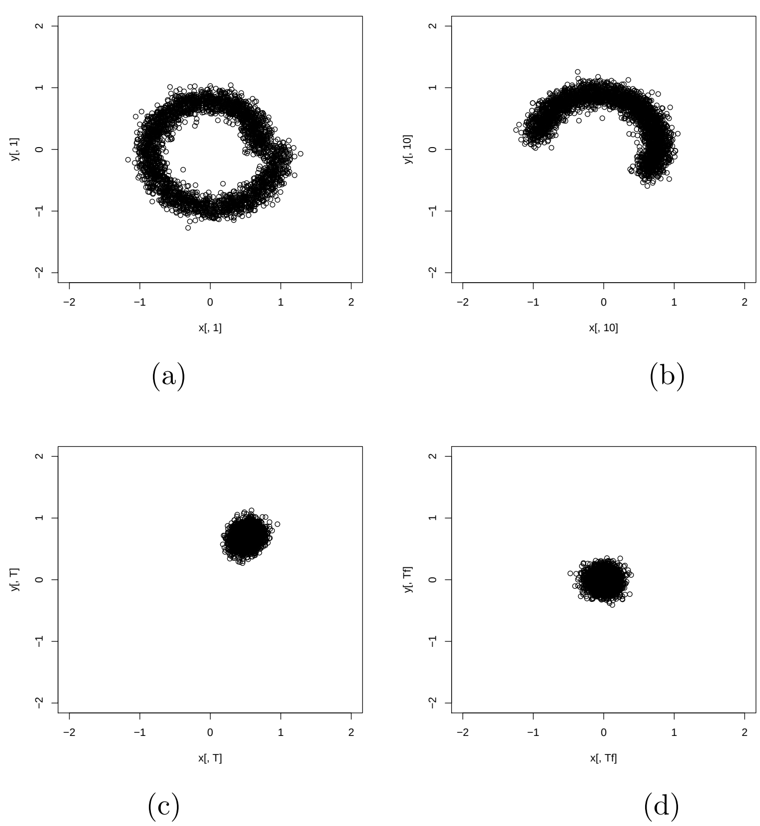


Figure 5: (a) Initial spiral pattern, with random "thickness" $2\delta = 0.2$. (b) After 10 cells. (c) After 20 cells. (d) After centering, and final filamentation; $\sigma_x = 0.014$, as discussed in the text.

Taking this particular example one step further, we estimate the expected beam intensity under the "ideal" conditions of our present analysis. With zero vertical emittance, the intensity might be expected to be proportional to the horizontal phase space area. Taking our spiral pattern given in Eq. 5, the arc length of the spiral is

$$\begin{aligned}\int ds &= \int_0^{2\pi} \sqrt{r^2 + (dr/d\theta)^2} d\theta = \int_0^{2\pi} \frac{1 + 16\pi^2(1+\theta/2\pi)^2}{32\pi^2(1+\theta/2\pi)}\, d\theta \\ &= \int_{4\pi}^{8\pi} \frac{1+u^2}{8\pi u}\,\frac{du}{2} = \frac{1}{16\pi}[\ln 2 + 24\pi^2] \\ &= (0.752)\cdot 2\pi.\end{aligned}$$

For an average thickness of 0.2 unit the area of this pattern would be approximately $2\pi(0.75)(0.2) = 0.3\pi$ sq. units. This is to be compared with an initial distribution which is uniform out to radius 1 unit, which has area $\pi$ sq. units. Thus, the spiral pattern would have about 1/3 the intensity of the uniform circular distribution. However, the former pattern would produce a beam, after flattening and unwinding, with variance of 0.014 sq. units, while the latter would have variance 0.25 sq. units. Thus, though the intensity is less, the beam brightness is enhanced by a factor of $(0.3/0.014)/(1.0/0.25) \approx 5$.

## 3 A Nonlinear FODO Channel

Suppose the nonlinear tune shift is generated by octupoles situated between the quadrupoles of a FODO channel. If the amplitude of the motion through the channel is $a$ at the focusing quad locations (where $\beta(s) = \hat{\beta}$), then the de-tuning term through the channel given by Eq. 4 can be written as

$$\nu - \nu_0 = \nu'' a^2 = \frac{3}{8\pi}\left(\frac{B_t}{B_s}\right)\left(\frac{\ell_o}{R}\right)\left(\frac{\hat{\beta}}{\beta_0}\right)\left\langle\left(\frac{\beta}{\hat{\beta}}\right)^2\right\rangle\left(\frac{a}{R}\right)^2$$

where $B_t$ is the pole tip field of the octupole magnets of total length $\ell_o$ within each complete cell, and $R$ is the pole tip radius.

To set the scale of a device to generate a small emittance flat beam, we consider parameters close to those of the Fermilab A0/FNPL photo-injector and beam line. Consider an electron with momentum 15 MeV/$c$ entering our FODO channel. The beam size leaving the Fermilab solenoid is on the scale of 5 mm or less. Suppose we let the channel be approximately 1-2 m long in total, and made up of 20 cells, each with a zero amplitude tune of 1/4. The amplitude function at the focusing magnets would be $\hat{\beta} \approx$ 0.085-0.17 m. For 15 Mev/$c$ electrons, $B\rho = 0.05$ T-m and so for a solenoid field of strength 0.1 T, as in the Fermilab set-up, we have $\beta_0 = 1$ m. (An appropriate optical matching section is required between the solenoid and the channel.) Pole tip fields of about 0.5-1.0 T at a radius of $R \approx 10$ mm, easily obtainable with permanent magnet materials, could be used.[6]

As a numerical example, Table 1 provides a list of general parameters for a 1.4 m long non-linear FODO channel for a 15 MeV electron beam. The channel is made up of magnetic elements with fields less than 1 T, which could presumably use permanent magnet technology. The 5 mm spiral at the solenoid becomes a 1.7 mm spiral at the entrance to the channel, and the magnet pole tip radius is taken to be about twice that. The maximum pole tip field in the device is about 0.5 T in the octupole magnets and 0.75 T in the quadrupole magnets.

| Parameter | Value | Unit | Note |
|---|---|---|---|
| SOURCE | | | |
| $p_e =$ | 0.015 | GeV/$c$ | electron momentum |
| $B\rho =$ | 50.0 | T-mm | magnetic rigidity |
| $B_s =$ | 0.1 | T | solenoid field strength |
| $k =$ | 1.0 | 1/m | coupling strength |
| $\beta_0 =$ | 1.0 | m | characteristic amplitude function |
| $a_0 =$ | 5 | mm | outer spiral amplitude at solenoid |
| QUADS | | | |
| $N_{cells} =$ | 20 | | Number of cells in channel |
| $L =$ | 35 | mm | half-cell length |
| $F =$ | 24.7 | mm | focal length |
| $\hat{\beta} =$ | 0.12 | m | maximum cell amplitude function |
| $a =$ | 1.7 | mm | outer spiral amplitude @ F quad |
| $R_q =$ | 3 | mm | quad pole tip radius |
| $\ell_q =$ | 8 | mm | quad length |
| $B' =$ | 250 | T/m | quad gradient |
| $B_q =$ | 0.76 | T | quad pole tip field |
| $\nu =$ | 0.25 | | cell tune |
| $L_{total} =$ | 1.400 | m | total channel length |
| OCTUPOLES | | | |
| $R_o =$ | 3 | mm | octupole pole tip radius |
| $\ell_o =$ | 20 | mm | octupole length [$\ell_o = (L - \ell_q) \cdot 0.75$] |
| $B_o =$ | 0.46 | T | octupole pole tip field |
| $\nu'' =$ | 0.0337 | 1/mm$^2$ | |
| $r_1 =$ | 1.2 | mm | inner spiral radius in FODO, @F quad |
| $r_2 =$ | 1.7 | mm | outer spiral radius in FODO, @F quad |
| $\Delta\nu =$ | 0.050 | | tune spread across spiral |
| $N_{cells} \cdot \Delta\nu =$ | 1.0 | | |

Table 1: Example parameters of a 1.4 m long 20-cell non-linear FODO channel.

While the above discussion is simplistic in its approach, with effects such as space charge not taken into account, it does provide an interesting avenue for further investigation. Plans are being made to illuminate the cathode of the A0/FNPL photo-injector with a spiral pattern using a mask in the laser beam and to study the resulting phase space distribution after the flat beam transformer. A future non-linear FODO channel may be in the offing.

The author would like to thank Don Edwards and Helen Edwards for many enlightening discussions and encouragement.

## References


[1] D. A. Edwards, *et al.*, *Proc. of the XX International Linac Conference, Monterey, CA*, pp. 122-124 (2000).

[2] R. Brinkmann, *et al.*, *Phys. Rev. ST Accel. Beams* **4**, 053501 (2001).

[3] Y. Sun, "Angular Momentum-Dominated Electron Beams and Flat-Beam Generation," PhD Thesis, U. of Chicago (2005).

[4] D. A. Edwards, private communication.

[5] D. A. Edwards and M. J. Syphers, *An Introduction to the Physics of High Energy Accelerators* John Wiley & Sons, New York (1993).

[6] J. K. Lim, *et al.*, *Phys. Rev. ST Accel. Beams* **8** 072401 (2005).